UNL Putnam Exam Study Seminar

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Today, we will do some problems involving **Induction**!

Induction Principle: *Give $P(n)$, a property depending on a positive integer n,*

(i) if $P(n_0)$ is true for some positive integer n_0, and

(ii) if for every $k \geq n_0$, $P(k)$ true implies $P(k+1)$ true,

then $P(n)$ is true for all $n \geq n_0$.

Some (possibly) helpful properties for today:

- Trig angle addition identities:

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

- A useful inequality: For all $x \in \mathbb{R}$: $e^x \geq x + 1$
- Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

It helps to brush up on useful properties prior to Putnam Exam! One strategy is to learn how to derive those properties (so that in case you forget some, you can still come up with them).

Problem 1. Show that for any $n \in \mathbb{N}$,

(a)

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b)

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

Problem 2. Prove that for any real numbers $x_1, x_2, \ldots, x_n$, $n \geq 1$,

$$|\sin x_1| + |\sin x_2| + \cdots + |\sin x_n| + |\cos(x_1 + x_2 + \cdots + x_n)| \geq 1.$$

Problem 3. Let $n \geq 6$ be an integer. Show that

$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$$