Thresholds in random graphs and the Kahn-Kalai Conjecture

Thanh Le

DisCo, 11/12/24

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$G(n,p)$: random graph on n vertices, each edge is present (independently) with probability p (Erdos-Renyi model).

Example: $G(4, 2/3)$

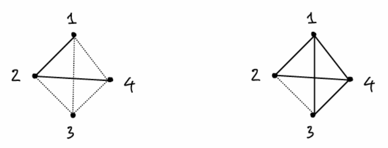

Figure: $\mu_{2/3}(G_1) = (2/3)^2(1/3)^4$, $\mu_{2/3}(G_2) = (2/3)^5(1/3)^1$

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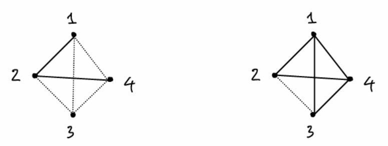

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Question: How big should p be so that $G(n, p)$ has certain property with high probability (whp)?

- E.g. containing a triangle? $p = w(1/n)$.

Property

A *property* $\mathcal{F}$ is a set of graphs that satisfy some conditions.

Monotone property: property that is closed under adding edges. E.g.

- Monotone properties: containing a triangle ($\mathcal{F}_{K_3}$); containing a perfect matching ($\mathcal{F}_{\text{perfect matching}}$)
- Non-monotone properties: containing an isolated vertex

Critical probability

Monotone property $\mathcal{F}$ in $G(n,p)$. $\mu_p(\mathcal{F})$ is strictly increasing wrt p with $\mu_0(\mathcal{F}) = 0$ and $\mu_1(\mathcal{F}) = 1$.

The *critical probability* of $\mathcal{F}$:

$$p_c(\mathcal{F}) = \{p : \mu_p(\mathcal{F}) = 1/2\}$$

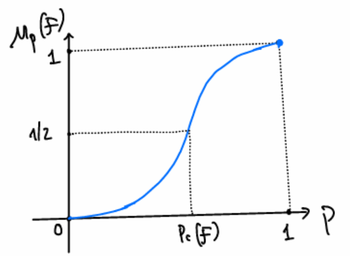

Figure: Graph of $\mu_p(\mathcal{F})$ with respective to p

Threshold function $p_t(n)$:

1. If $p = w(p_t(n))$, then $\mu_p(\mathcal{F}) = 1 - o(1)$.
2. If $p = o(p_t(n))$, then $\mu_p(\mathcal{F}) = o(1)$.

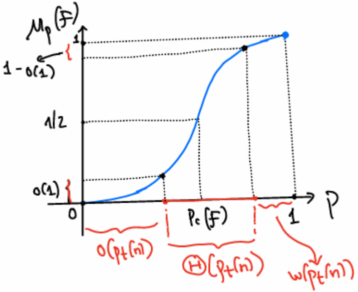

Figure: Illustration for threshold functions

Threshold phenomena

Theorem (Bollobas-Thomason 1987). Every monotone property has a threshold function; moreover one can take $p_c(\mathcal{F})$ to be this threshold function.

GOAL: find the asymptotic order of $p_c(\mathcal{F})$. Methods prior to Kahn-Kalai Conjecture:

- Lower bound: first-moment method
- Upper bound: second-moment method, hitting-time result.

Example on identifying p_c (i.e. the threshold)

Example: $p_c(\mathcal{F}_{K_3}) = \Theta(1/n)$.

Proof outline.

First-moment method: Let X be the random variable for number of triangles in $G(n,p)$.

$$\mathbb{E}X = \binom{n}{3} p^3$$

$$\mu_p(\mathcal{F}_{K_3}) = \mu_p(X \geq 1) \leq \frac{\mathbb{E}X}{1} \sim \frac{n^3p^3}{6} = o(1)$$

when $p = o(1/n)$. So, $p_c(\mathcal{F}) = \Omega(1/n)$.

Second-moment method: when $p = w(1/n)$: $\mathbb{E}X$ is big. Use $Var(X)$ to assert that $\mu_p(X = 0) = o(1)$, and hence $\mu_p(\mathcal{F}_{K_3}) = 1 - o(1)$. □

Kahn-Kalai Conjecture - Motivation and setup

Motivation: it's often straightforward to give a lower bound for $p_c(\mathcal{F})$. Kahn and Kalai (2006) conjectured that the best possible easy lower bound is within a $\log n$-factor of the truth.

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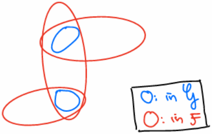
General setting:



- Monotone property $\mathcal{F} \subseteq \{0,1\}^N$.
  - E.g. for $G(n,p)$: $N = \binom{n}{2}$.
- A cover $\mathcal{G}$ for $\mathcal{F}$: $\mathcal{G} \subseteq \{0,1\}^N$ such that $\forall T \in \mathcal{F} \, \exists S \in \mathcal{G} : S \subseteq T$.
  - E.g. for $\mathcal{F}_{K_3}$, $\mathcal{G}$ can be {graphs with a triangle and no other edges}.
- $L(\mathcal{F})$: be the maximum size of a minimal element of $\mathcal{F}$.
  - E.g. for $\mathcal{F}_{K_3}$, $L(\mathcal{F}_{K_3}) = 3$.

The easy lower bound: Expectation threshold

A property $\mathcal{F} \subseteq \{0,1\}^N$ is *p-small* if there exists a cover $\mathcal{G}$ such that

$$\sum_{S \in \mathcal{G}} p^{|S|} \leq \frac{1}{2}$$

$\mathcal{F}$ is p-small then:

$$\mu_p(\mathcal{F}) \leq \sum_{S \in \mathcal{G}} \sum_{T : S \subseteq T} \mu_p(T) = \sum_{S \in \mathcal{G}} p^{|S|} \leq \frac{1}{2},$$

so $p \leq p_c(\mathcal{F})$.

The *expectation threshold* of $\mathcal{F}$ is defined as

$$p_E(\mathcal{F}) := \max_p (\mathcal{F} \text{ is } p\text{-small})$$

Key: $p_E(\mathcal{F}) \leq p_c(\mathcal{F})$.

Kahn-Kalai Conjecture, now Park-Pham Theorem

Theorem (Park-Pham 2022). There exists an absolute constant K so that for every monotone $\mathcal{F}$,

$$p_c(\mathcal{F}) \leq K \cdot p_E(\mathcal{F}) \cdot \log L(\mathcal{F}).$$

Apply Park-Pham Theorem

To identify $p_E(\mathcal{F})$: use a fractional version of Kahn-Kalai and linear programming duality.

Definition. Let $\mathcal{F} \in \{0,1\}^N$. A probability measure ν supported on $\mathcal{F}$ is *p-spread* if for all $S \in \{0,1\}^N$,

$$\sum_{T \supseteq S} \nu(T) \leq 2p^{|S|}$$

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Theorem (Frankston, Kahn, Narayanan, and Park, 2019). There is an absolute constant K so that the following is true. Let $\mathcal{F}$ be a monotone property that supports a p-spread probability measure ν. Then

$$p_c(\mathcal{F}) \leq K \cdot p \cdot \log L(\mathcal{F})$$

Example on $\mathcal{F}_{\text{perfect matching}}$

Claim: $p_c(\mathcal{F}_{\text{perfect matching}}) = \Theta(\frac{\log n}{n})$.

Proof.

Upper bound: Park-Pham: define ν that is uniform on all perfect matchings on n vertices (and 0 elsewhere). For $S \in \{0,1\}^N$:

- If S is not a matching: then $\sum_{T \supseteq S} \nu(T) = 0$.
- If S is a matching: then

$$\sum_{T \supseteq S} \nu(T) = \text{pm}(K_{n-2k}) \frac{1}{\text{pm}(K_n)}$$

$$= \frac{(n-2k)!}{2^{n/2-k}(n/2-k)!} \frac{2^{n/2}(n/2)!}{n!} \leq \left(\frac{e}{n}\right)^k$$

Example on $\mathcal{F}_{\text{perfect matching}}$ (cont)

Claim: $p_c(\mathcal{F}_{\text{perfect matching}}) = \Theta(\frac{\log n}{n})$.

Proof (cont.).

Then, ν is $\frac{e}{n}$-spread $\Longrightarrow p_c(\mathcal{F}) = O(\frac{\log n}{n})$.

Lower bound: First-moment for μ_p(having isolated vertex), we get that $p_c(\mathcal{F}) = \Omega(\frac{\log n}{n})$. □

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Lower bound: First-moment for μ_p(having isolated vertex), we get that $p_c(\mathcal{F}) = \Omega(\frac{\log n}{n})$. □

Comments:

- People used non-uniform ν for harder problems, e.g. Latin squares, containment of a square of Hamilton cycle.

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Thank you!!